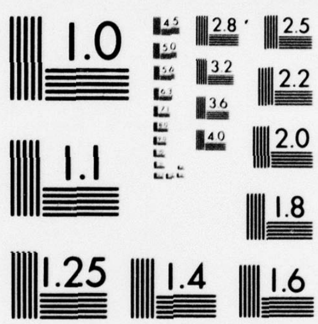
AD-A066 044 STEVENS INST OF TECH HOBOKEN N J DEPT OF MECHANICAL --ETC F/G 19/4
EXTENDING FINITE ELEMENT METHODOLOGY FOR A CLASS OF IMPACT PROB--ETC(U)
FEB 79 A T CHANG DAAG29-78-G-0015

UNCLASSIFIED ARO-15522.1-R-E NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD A0 66044

DDC FILE COPY

LEVEL

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 15522.1-R-E (18) ARO	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Extending Finite Element Methodology for a Class of Impact Problems.	5. TYPE OF REPORT & PERIOD COVERED: Final Report, 1 December 1977 to 31 December 1978
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Dr. Anthony T. Chang	8. CONTRACT OR GRANT NUMBER(s): DAAG 29-78-G-0015
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Stevens Institute of Technology, Hoboken, N. J. 07030	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: U. S. Army Research Office, P. O. Box 12211, Research Triangle Park, NC 27709	12. REPORT DATE: Feb 1979
	13. NUMBER OF PAGES: 31
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Projectiles, Targets, Finite Element Methodology, Impact Problems, Solid Media, Surface Wanes, Nonlinear Material Problems, High Speed Impact

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

High speed impact of an object on solid media is a complicated phenomena which is influenced by variables such as material properties, impact velocity, projectile shape, target support position and relative dimensions of the target and projectile. This research has concentrated on a class of impact problems whose angle of impact is so shallow that the target impact disturbance might be considered as a surface wave problem. Most of research effort was spent in doing nonlinear material problems. The theoretical background for the pressure distribution developed during impact is given in the last chapter. Unfortunately no "numerical experiments" were performed to verify the theory presented.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

334 580 LD

Extending Finite Element Methodology for a Class of Impact Problems

Contract No. DAAG 29-78-G-0015
Army Research Office

By

Dr. Anthony T. Chang
Dept. of Mechanical Engineering
Stevens Institute of Technology
Hoboken, N.J. 07030

Contents

1. Introduction

High speed impact of an object on solid media is a complicated phenomena which is influenced by variables such as material properties, impact velocity, projectile shape, target support position and relative dimensions of the target and projectile. In this research work we have concentrated our effort in the investigation of a class of impact problems whose angle of impact is so shallow that the target impact disturbance might be considered as a surface wave problem.

The above assumption has certain advantage. The theory for the wave disturbance resulting from the application of a pressure distribution on the fluid surface is well known, Lamb [1]. Knowing the shape of the projectile the resulting pressure at the contact area can be determined. Under these pressure the dynamic behavior of the projectile can be computed by a step by step approach.

Our other goal is to simulate this class of impact problems using a general purpose finite element program. The NASTRAN program was selected because this general purpose program is most widely used by the Army ordnance engineering community, and the program has the capability of handling load matrices due to nonlinearity.

Before we could start to study the pressure developed during impact we had to be certain that NASTRAN can handle any nonlinear material problems.

Most of this first year of research on this project was spent in doing nonlinear material problems using NASTRAN. The theoretical background for the pressure distribution developed during impact is given in the last chapter. Unfortunately no "numerical experiments" were performed to verify the theory presented.

2. Nonlinear Material Problems using NASTRAN

NASTRAN is a finite element computer program for structural analysis that is based on the displacement approach. The program includes a limited capability for the solution of nonlinear elastic-plastic problems using Piecewise Linear Analysis. In version 16 of NASTRAN the elements that do have this capability are ROD, TUBE, BAR, and PLATE. In this report a Direct Iteration Method is described that allows the elastic-plastic analysis of structures that can be represented by combinations of elements contained in the NASTRAN element library, such as beams, rods, shear and twist panels, triangular and quadinlateral plates, conical and toroidal shells, solids of revolution, scalar elements, three dimensional elements, isoparametric elements, and constraint elements.

In the Piecewise Linear Analysis the load is applied in increments such that the stiffness properties can be assumed to be constant over each increment. The stiffness matrix for each increment is dependent on the current state of stress in the structural elements. The increments in displacements and stresses are accumulated to produce the final, nonlinear results. Since the algorithm assumes linearity between sequential loads, the results will depend on the user's choice of load increments. When the user selects large load increments and the material properties are changing rapidly, the results may be unacceptably inaccurate. If small load increments are used when the structure is nearly linear the solution will be more accurate but relatively costly. Based on the same approach Yang and Frederick [2] expanded the NASTRAN capability to include axisymmetric solid ring elements.

The Direct Iteration Method provides a very simple but approximate solution of an elastic-plastic problem. In brief, the procedure is as follows. Assume that all material is elastic, compute the structure stiffness matrix, apply the full load to the structure, and solve for displacements and stresses. Compute strains in each element. For those elements strained beyond their elastic limit,

approximately reduce their elastic stiffness, and compute a different structure stiffness matrix based on the modified elastic properties. Again apply the full load and compute displacements and stresses. Stop iteration when source convergence criteria is satisfied. The final set of displacements and stresses is the elastic-plastic solution of the structure.

The principal advantage of the Direct Iteration Method is its simplicity. However, there are several limitations. The method may not converge if plastic region is extensive. Possible elastic unloading of plastically deformed material is not accounted for. The flow rules of plasticity are ignored by the assumption that the solution is uniquely defined by whatever final load level is reached. It is known that the final state of stress and strain depends on how the stresses and strains develop during loading as well as how much load is applied. Another approximation is introduced by the application of deformation theory, which is exact only if all stresses increase proportionally, that is, if in reality, at any point and throughout loading, the resulting stress distribution does not depend on the order of applied loadings.

With the Direct Stiffness Method the strain energy in each element is always underestimated, therefore a higher stiffness is assumed for each element. For most small strain (in the order of 10^{-3}) problems this approximation is not critical as shall be demonstrated in the examples.

The elastic-plastic nonlinearity of a structural element is defined by the element material used. Any isotropic material may be made nonlinear by including a stress-strain table defining its extension test characteristics. This stress-strain table must define a nondecreasing sequence of both stresses and strains. Because the stiffness matrix for the first iteration step uses the elastic material coefficients, the initial slope should correspond to the defined Young's Modulus, E.

The stress-strain curve can be purely elastic or elastic-plastic. In the later case, the stress at which plastic yielding begins has to be given. For this field criterion the hypothesis of von Mises seems to be most widely accepted, i.e. that yielding starts when the effective or equivalent stress

$$\bar{\sigma} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2) + (\sigma_2 - \sigma_3) + (\sigma_3 - \sigma_1)} \qquad (1)$$

reaches the yield point of the material. In the above equation σ_1, σ_2 and σ_3 are the principal stresses.

Once the stress-strain relationship is defined the calculation may be described by the following steps:

1) After applying the full load to the structure the elastic stresses and strains are obtained. Strain energies U_i of the individual elements are then computed. In NASTRAN version 16 or later versions, strain energy can be requested as part of the output.

2) The equivalent strains of each element can be calculated

from the strain energy

$$\epsilon = \sqrt{\frac{2U_e}{EV}} \tag{2}$$

where V is the volume of the element. To identify the nonlinear elements these strains are compared with respect to the stress-strain curve.

3) Plastic elements are determined by calculating the equivalent stress of the elements. In all but the simplest elements the amount of plastic action is likely to vary from one point in the element to another. It is therefore necessary to select a point whose equivalent stress will represent the behavior of that element. In simple elements, the centroid is a good sampling point. In isoparametric elements the average of the Gauss points values may be chosen. The general rule is that sampling point should be placed where strains are likely to be computed most accurately.

4) For those elements that are nonlinear a different modulus of elasticity has to be determined from the stress-strain diagram. In a typical element, the first elastic solution establishes line OA. The corresponding strain $\bar{\epsilon}_1$

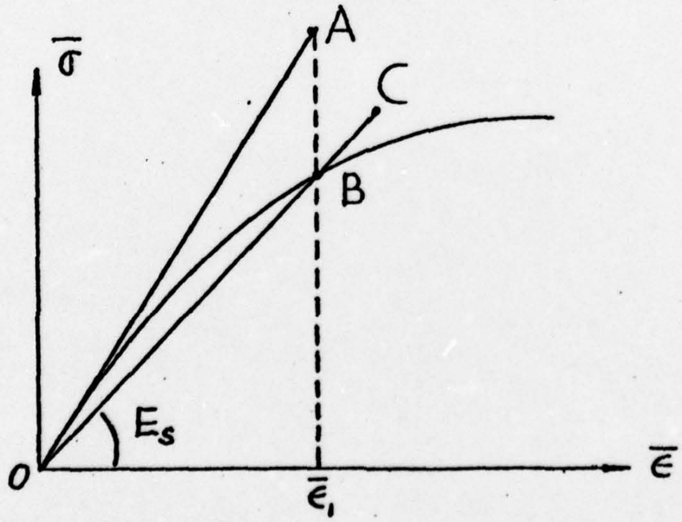


Figure 1: Direct Iteration in Stress-Strain Curve

is obtained from the strain energy. From the line OB a secant modulus

can be computed.

5) With the new secant modulus E_s an elastic analysis is again carried out to reach point C in the stress-strain diagram. If this element is in the plastic range, the Poisson's ration ν is to be equal to 0.5 (for solid elements, set $\nu = 0.499$). This is to take into account of the simplified flow rules.

6) Step 2 is repeated.

7) If there is plastic action step 3 is repeated

8) Step 4 is repeated, etc.

9) The calculation is continued until the stresses or strains do not change beyond certain acceptable percentage points.

3. Modifications

In step 5) of each iteration a complete reassembly and reduction of the structure stiffness matrix is required. To avoid this inefficiency, all nonlinearity may be imposed as additional loadings. If $[k]$ is the original stiffness matrix of an element, the reduced element matrix is

$$[k_r] = \frac{E_s}{E}[k] = (1 - \frac{E-E_s}{E}) = [k] - [k_m] \qquad (3)$$

where $[k_r] = (1 - \frac{E_s}{E})[k] = a[k]$

Thus the equilibrium equation at any iteration step n may be written as

$$[K]\{D\}_n = \{R\} + [K_m]_{n-1}\{D\}_{n-1} \qquad (4)$$

where $[k]$ is the original assembled structure stiffness matrix

$\{D\}_n$ is the displacement at iteration step n

$\{R\}$ is the total applied load

$[k_m]_{n-1}$ is the assembled structure stiffness matrix based on at step n-1

$\{D\}_{n-1}$ is the displacement matrix at step n-1.

In structure stiffness matrix $[K_m]_{n-1}$ only the modified elements are assembled.

Another advantage of this approach is that it offers the possibility of under relaxation. Unrelaxation is accomplished in the selection of previous nodal displacement $\{D\}_{n-1}$ at step n in the above equation. The displacement can be given as

$$\{D\}_{n-1} = (1-\beta)\{D\}_{n-2} + \beta\{D\}_{n-1} \tag{5}$$

where $0<\beta<1$. For β=1 there is no relaxation. Any other β value the load due to modified secant modulus is reduced at each iteration step. The purpose is to ensure that the resulting stresses converge after each iteration. Convergence is

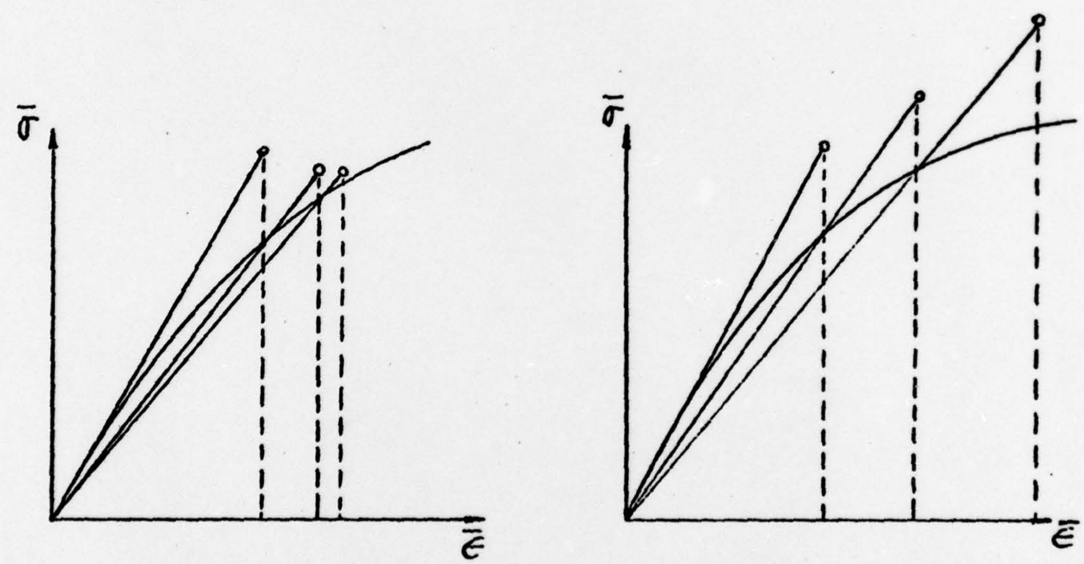


Figure 2: Convergence and Divergence

illustrated in the left hand figures. If there is divergence as shown in the right hand figure, no solution can be obtained.

There is another way to avoid divergence of the solution. This is to apply a fraction of the load first. After completing the iteration process, modulus of elasticity are obtained for the next increased loading. This will avoid reading a point far away from the stress-strain curve at the first iteration step, and yielding may occur in elements which later in the process must be unloaded. It is assumed that with partial loading, the equivalent stresses in some elements do exceed the yield point.

Partial loading is often necessary for problems where many elements are in the plastic range. However, satisfactory results can be obtained even with rather coarse load steps. This point is demonstrated in the examples given in this report.

4. Examples

Case I. Five-Rod Truss

This example was selected because Yang and Frederick had worked out this problem using both their incremental approach and the price-wise incremental option available in NASTRAN (Rigid-Format 6).

The truss is loaded with 100,000 pounds as shown in Figure 3. The elastic-plastic material properties for each member are given in Figure 4.

The forces in this truss are statically determinate, therefore the stresses do not change with different modulus of elasticity. In order to plot the complete behavior of the members at different bad level, two loads (80,000 and 100,000 lb.) were applied to the truss. Numerical results for direct iteration analysis of the truss are listed in Table 1.

Table 1:

Direct Iteration Analysis of Five-Rod Truss

Load	MODULUS OF ELASTICITY $E_1=E_2=E_3=E_4$			Vertical Displacement of Node 5	Stresses $\sigma_1=\sigma_2=\sigma_3=\sigma_4$	$\sigma_5=\sigma_6$
80,000	$10(10)^6$	$40(10)^7$	$10(10)^6$	$2.66(10)^{-2}$	$5.657(10)^4$	$8(10)^4$
80,000	$4.88(10)^6$	$21.3(10)^7$	$10(10)^6$	$5.39(10)^{-2}$	$5.657(10)^4$	$8(10)^4$
100,000	$2.82(10)^6$	$17.4(10)^7$	$10(10)^6$	$1.12(10)^{-1}$	$7.071(10)^6$	$10(10)^4$

The vertical displacements of joints 2 and 5 are plotted in Figure 5. The points used to determine the different modulus of elasticity are indicated in Figure 6. Since the stresses do not change for the same applied load there is no need for iteration in this sample. There is no difference between the results obtained by direct iteration and the results obtained by Yang & Frederick using incremental approaches. The final iteration run on the computer is listed at the back of this report.

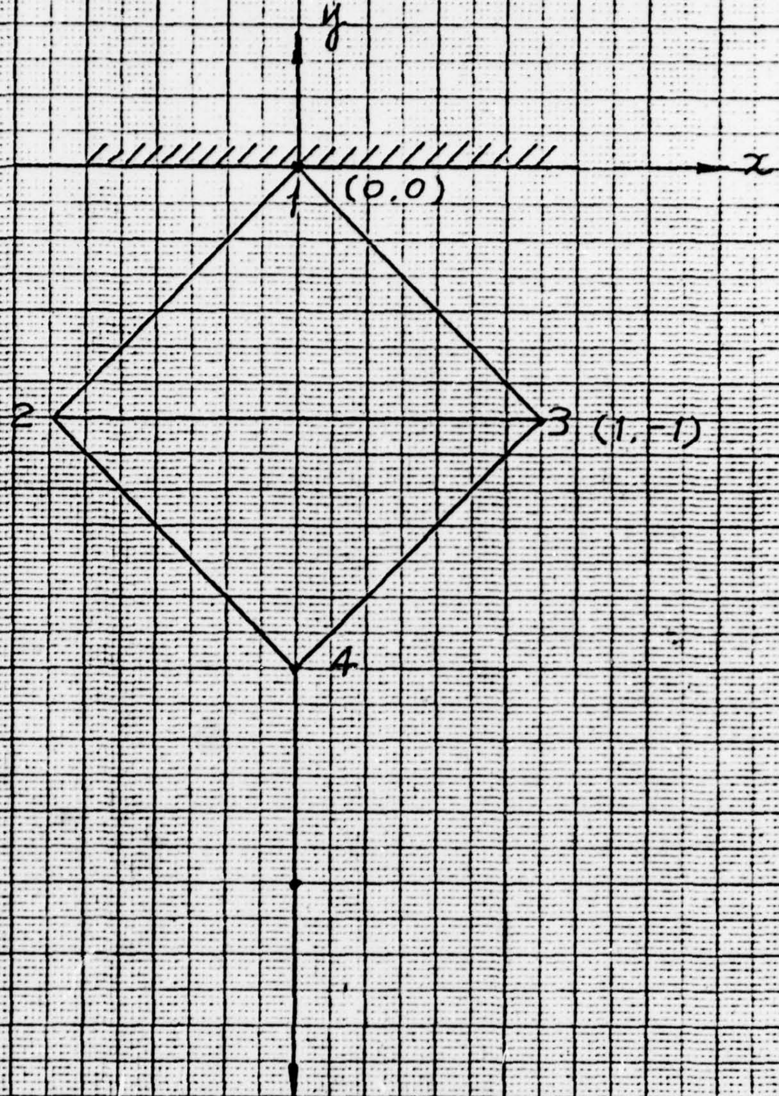


Figure 3: Truss, Five-Element

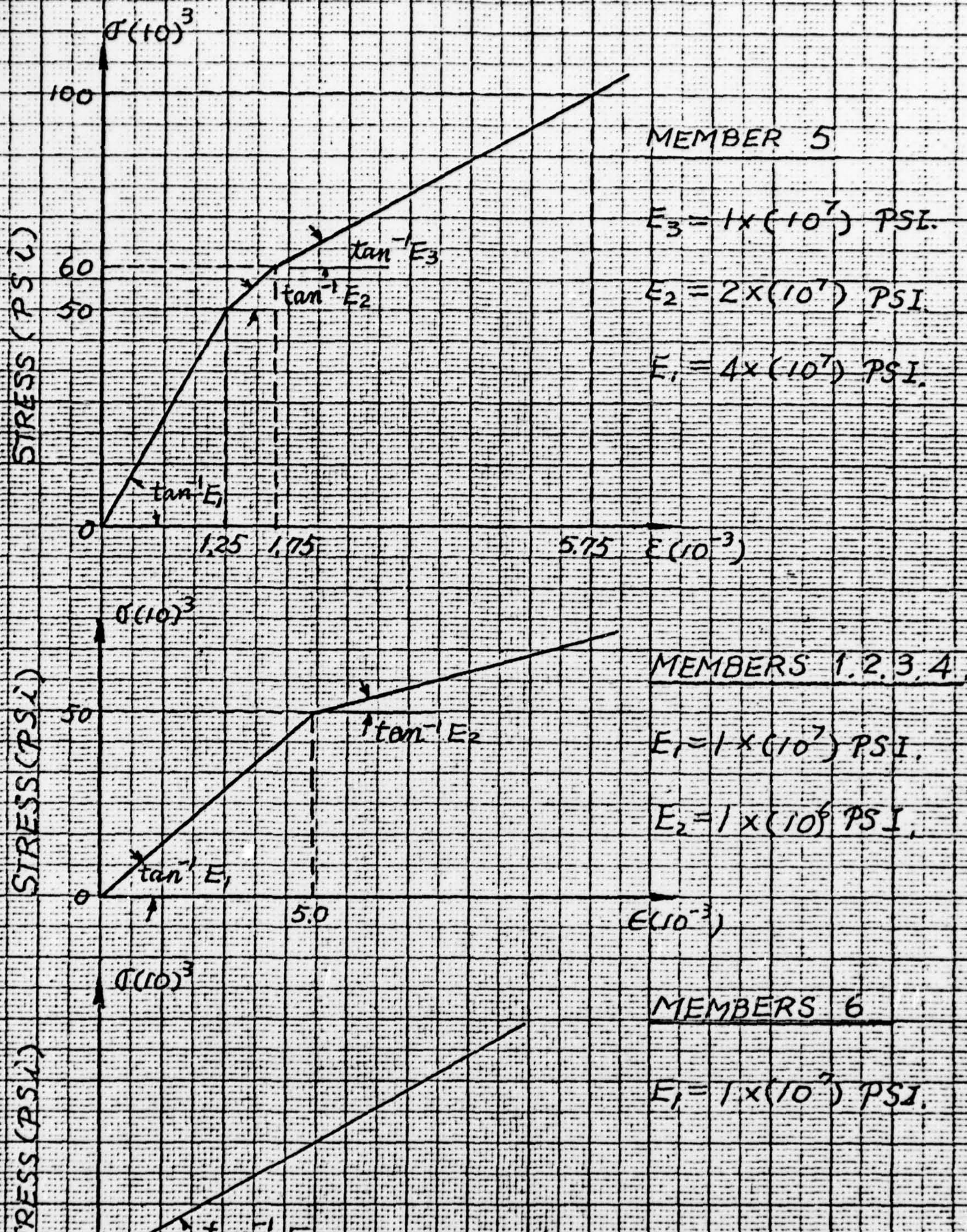


Figure 4. STRESS – STRAIN CURVES.

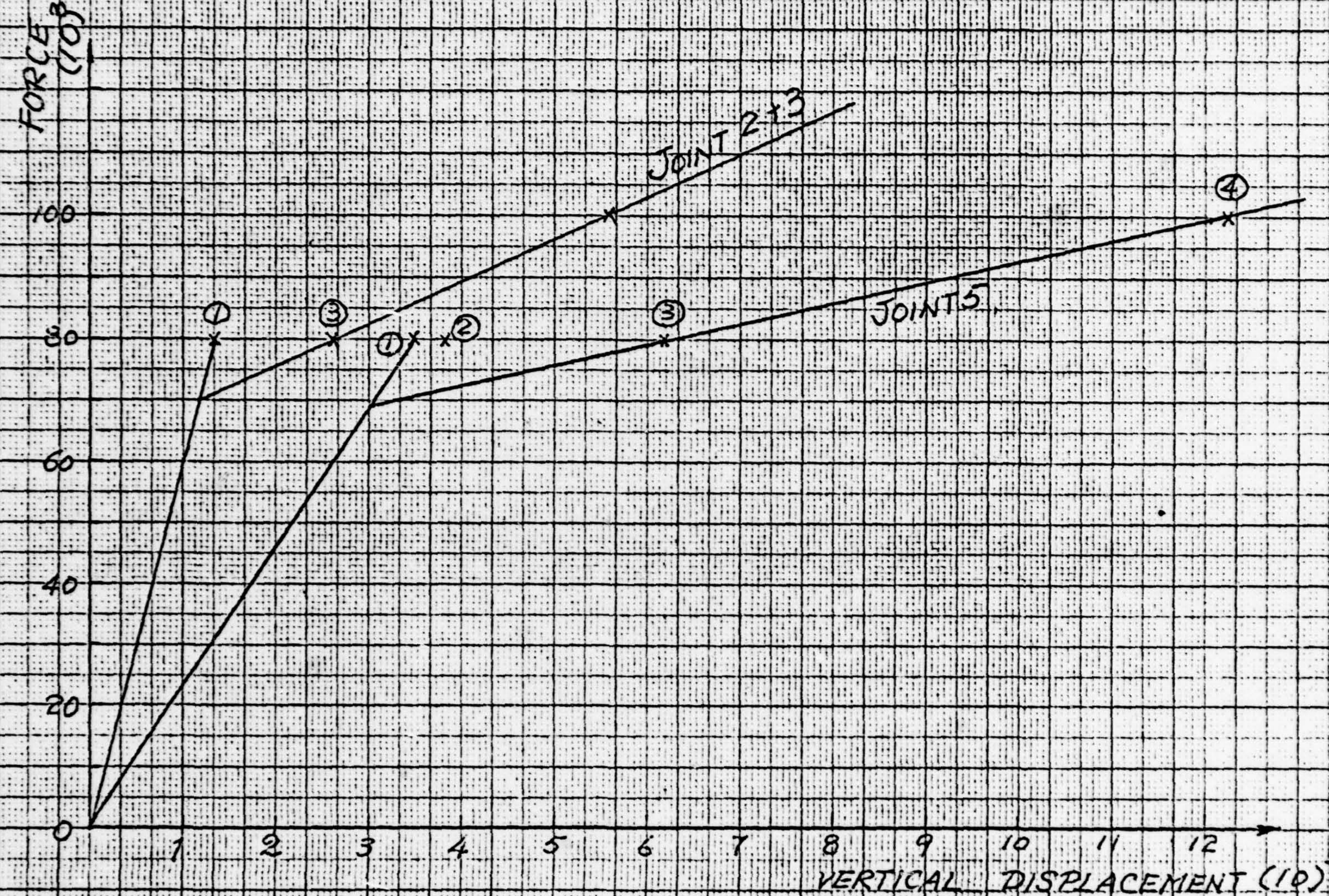


Figure 5: VERTICAL DISPLACEMENT

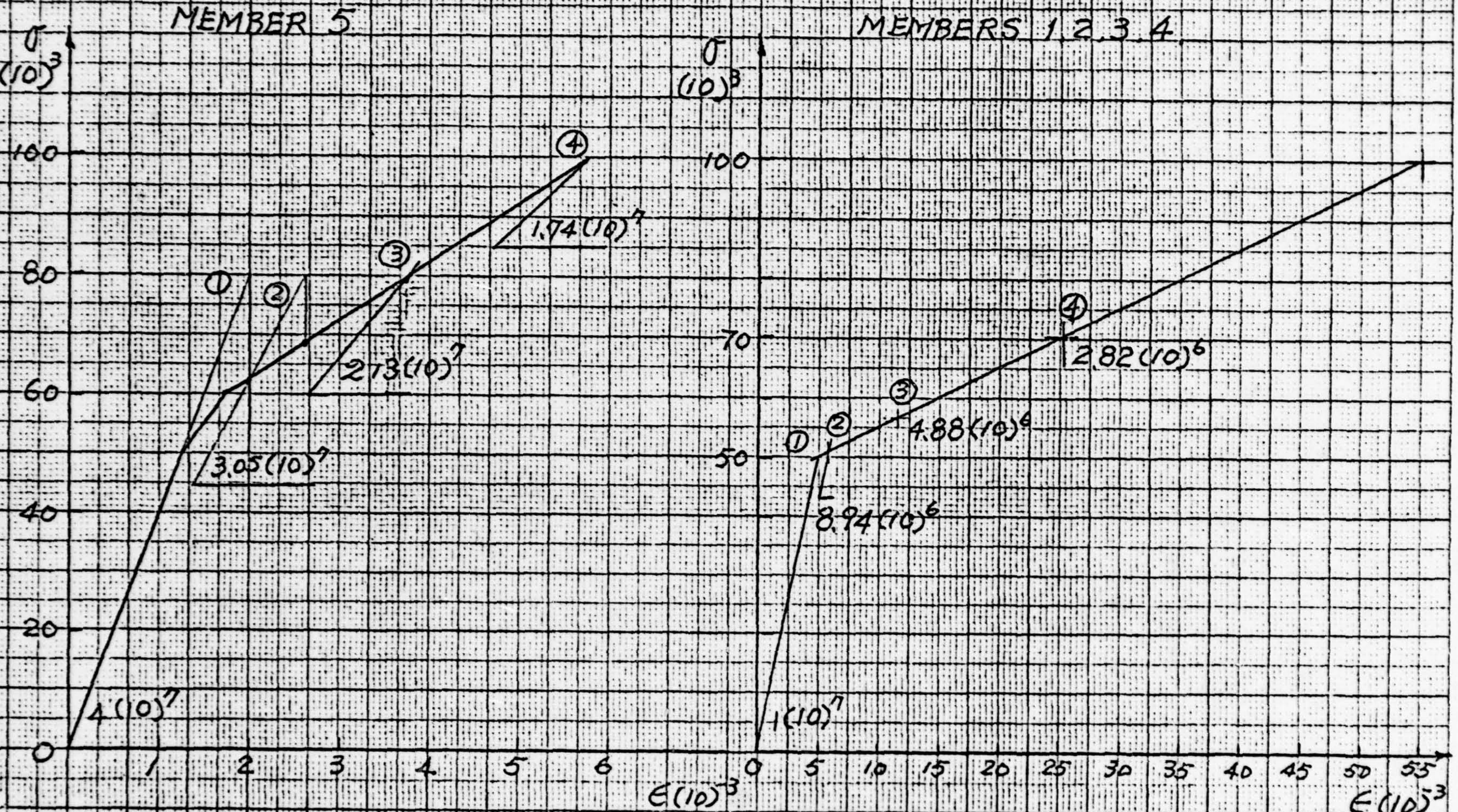


Figure 6: STRESSES IN MEMBERS FIVE-ROD TRUSS AUG. 10, 1978

Case II. Three-Rod Truss

The simple Three-Rod Truss (see Figure 7) is statically indeterminate. This problem is worked out by Martin and Carey [3] using a nine step incremental approach. The Ramberg-Osgood description of the uniaxial stress-strain curve was used to represent the elastic-plastic material behavior:

$$\epsilon = \frac{\sigma}{E} + \frac{3}{7}\frac{\sigma_{YP}}{E}\left(\frac{\sigma}{\sigma_{YP}}\right)^n$$

where E = initial elastic modulus

σ_{YP} = yield stress

n = integer specifying strain-hardening characteristics of the material. For this truss problem, $E = 11.6(10)^6$ psi, $\sigma_{YP} = 43000$ psi, and $n=11$. The stress-strain curve is plotted in Figure 8.

Applying the load of 120000 lb., six iterations were used to approximate the solution. The secant modulus of elasticity of each iteration step is indicated in Figure 8. The results are listed in Table 2. It compares favorabilly with solutions given by the incremental method. The final iteration run is listed at the back of this report.

Table 2: Direct Iteration Method of Three-Rod Truss

	SECANT MODULUS OF ELASTICITY		DISPLACEMENT	STRESSES	
Iteration	$E_1=E_3$	E_2	V_1	$\sigma_1=\sigma_3$	σ_2
1	$1.16(10)^7$	$1.16(10)^7$	0.606×10^{-2}	35147	70294
2	$1.106(10)^7$	$7.59(10)^6$	0.779×10^{-2}	43054	59113
3	$1.04(10)^7$	$6.26(10)^6$	0.818×10^{-2}	45832	55184
4	$9.24(10)^6$	$5.53(10)^6$	0.995×10^{-2}	45956	55008
5	$8.55(10)^6$	$4.98(10)^6$	1.088×10^{-2}	46527	54200
6	$8.42(10)^6$	$4.63(10)^6$	1.134×10^{-2}	47733	52495
Result from Incremental Approach[2]			1.42×10^{-2}	47916	52192

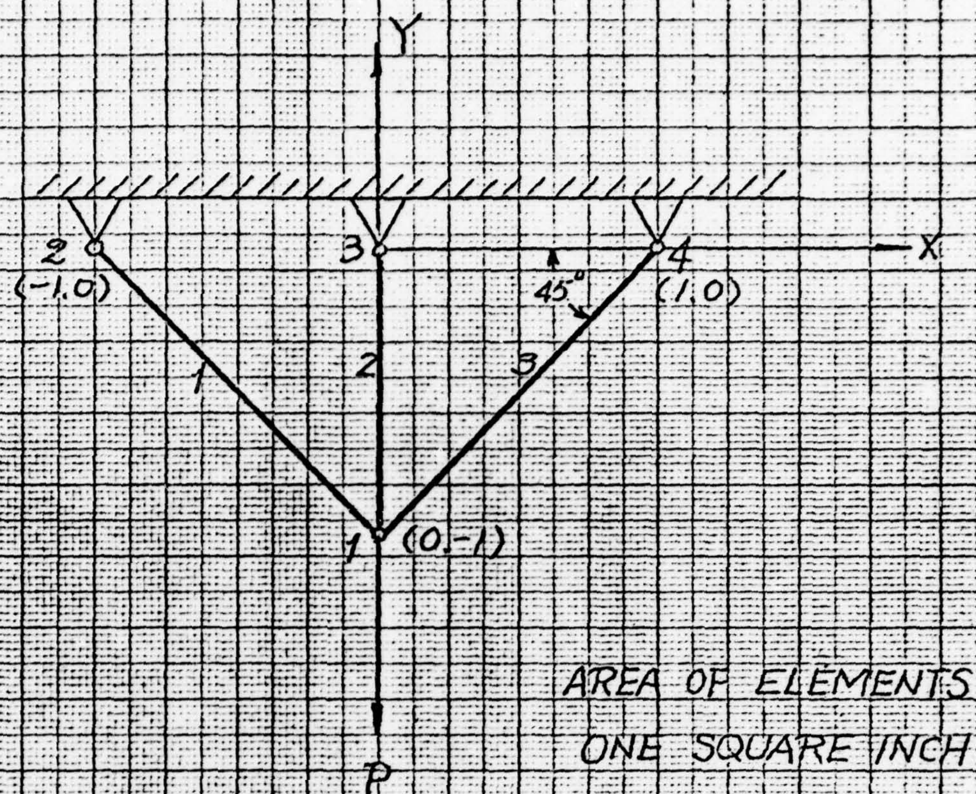


Figure 7: THREE ROD TRUSS

THREE ROD TRUSS

$$\epsilon = \frac{\sigma}{E} + \frac{3}{7}\,\frac{\sigma_{YP}}{E}\left(\frac{\sigma}{\sigma_{YP}}\right)^{n}$$

$$= \frac{\sigma}{11.6\times10^{6}} + \frac{3}{7}\,\frac{43500}{11.6\times10^{6}}\left(\frac{\sigma}{43500}\right)^{11}$$

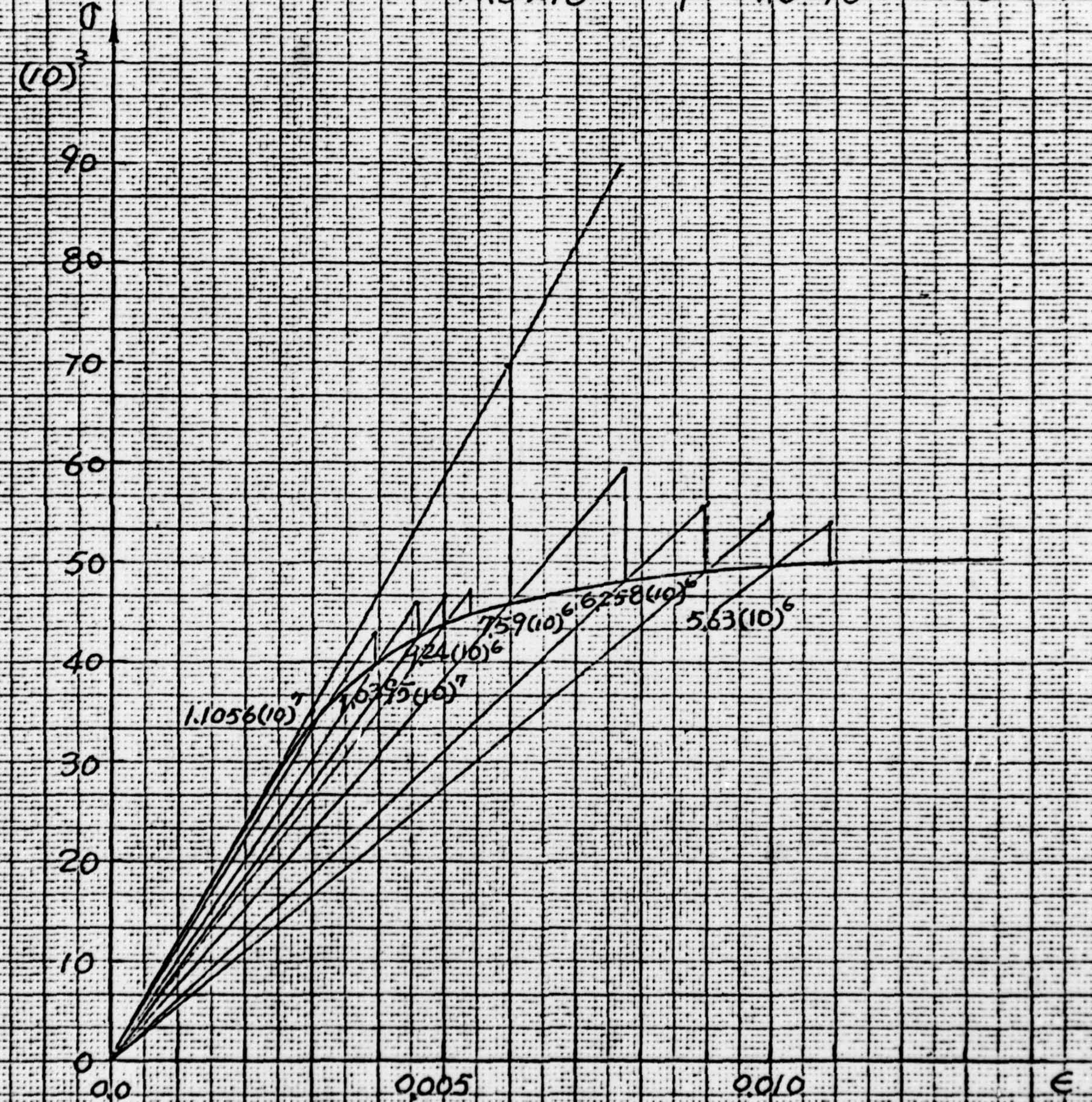


Figure 8: STRESS — STRAIN DIAGRAM

Case III. Thick-Walled Cylinder

The thick-walled cylinder under constant internal pressure is another numerical example calculated by Yang and Frederick [2] using the manual incremental approach. With the direct iteration method 80 axisymmetric rectangular solid elements were used to model the cylinder (Fig. 9). The stress-strain relationship of the material is given in Fig. 10. Starting with modulus of elasticity equaling $30(10)^6$ psi the strain energies of the elements are computed. Using the formula of equations (2) the equivalent strain of each element was determined. With this strain energy, secant modulus of elasticity was obtained from the stress-strain diagram. (Fig. 10). If the strain was in the plastic range, Poisson's ratio is set to be 0.49.

Two internal pressure loads 54,562 psi and 110,570 psi were applied. For the lower pressure only one iteration step beyond the original run was made. The results are shown in Figures 11, 12 and 13. Figures 14, 15 and 16 show the results of the higher pressure case after three iteration runs. The theoretical solutions were taken from Yang and Frederick. The agreement is good enough for most engineering applications.

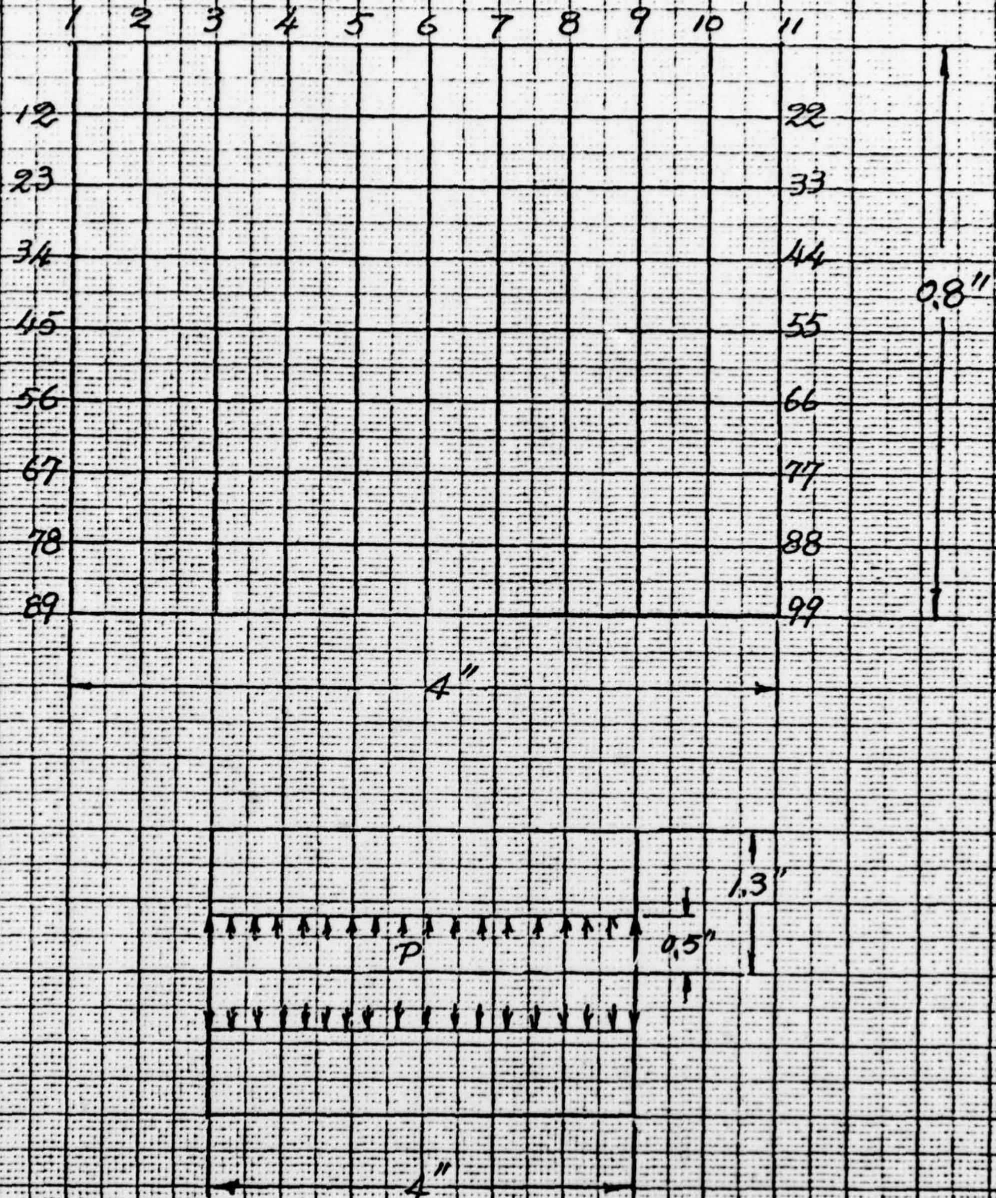


Figure 9. NASTRAN MODEL,
THICK WALL CYLINDER.

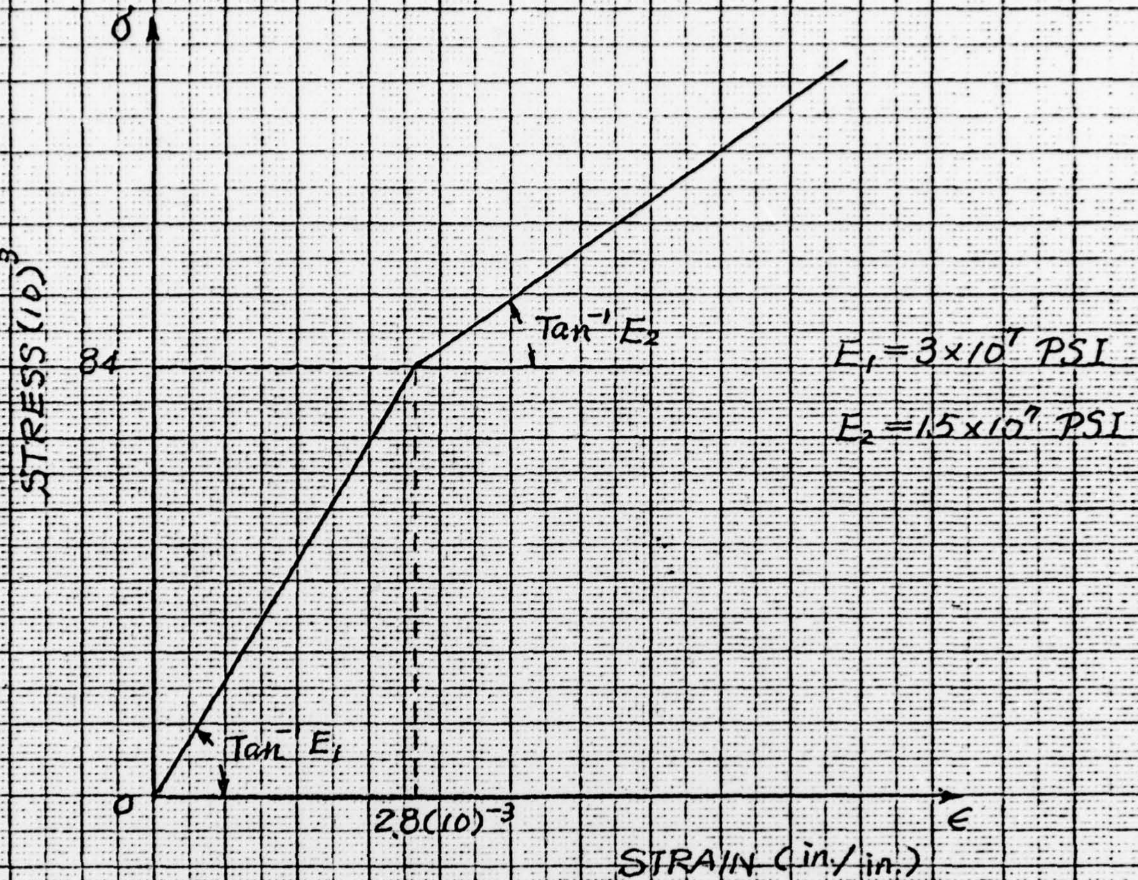


Figure 10. STRESS-STRAIN CURVE, THICK WALL CYLINDER

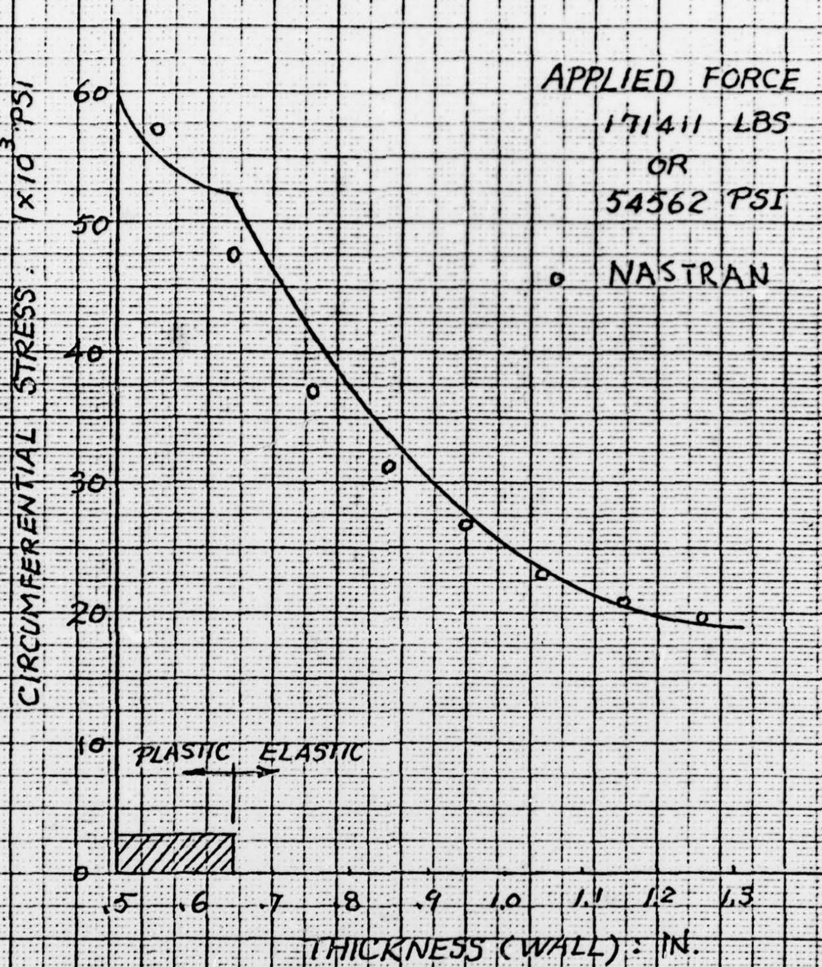


Figure 11: CIRCUMFERENTIAL STRESS VS THICK WALL CYLINDER, AT 54.562 PSI. APPLIED FORCE.

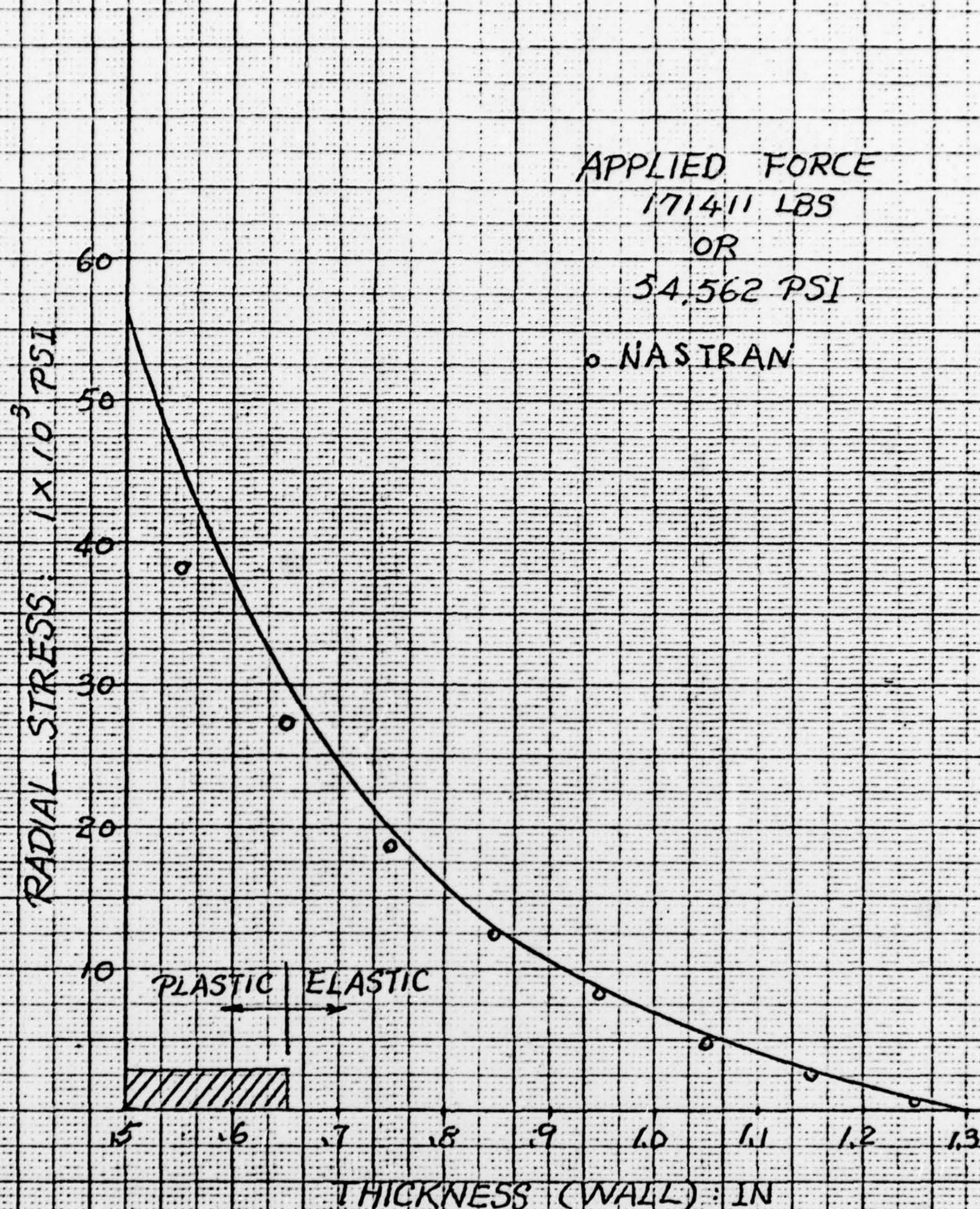


Figure 12. RADIAL STRESS VS WALL THICKNESS, THICK WALL CYLINDER, AT 54,562 PSI APPLIED FORCE.

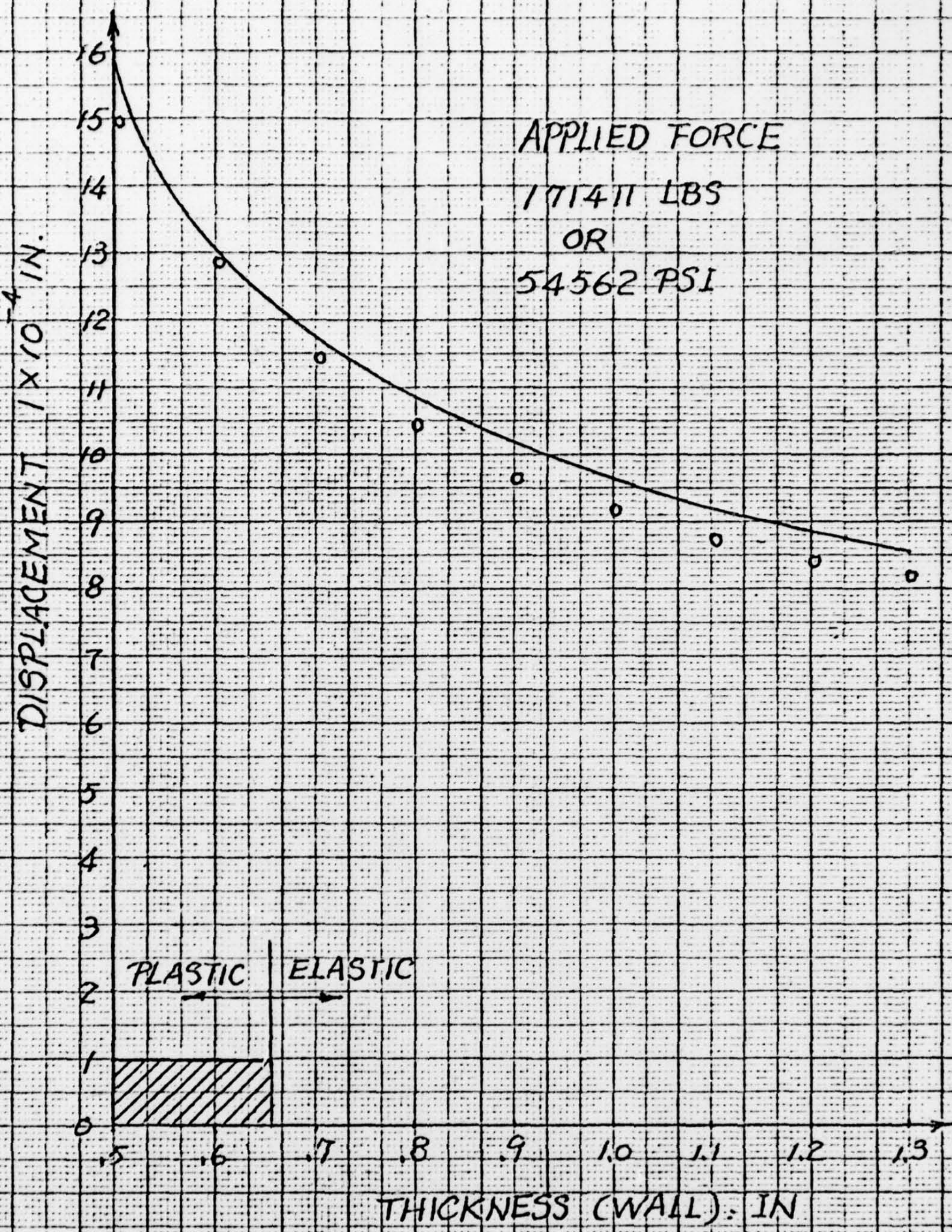


Figure 13. RADIAL DISPLACEMENTS, THICK WALL CYLINDER, AT 54,562 PSI APPLIED FORCE.

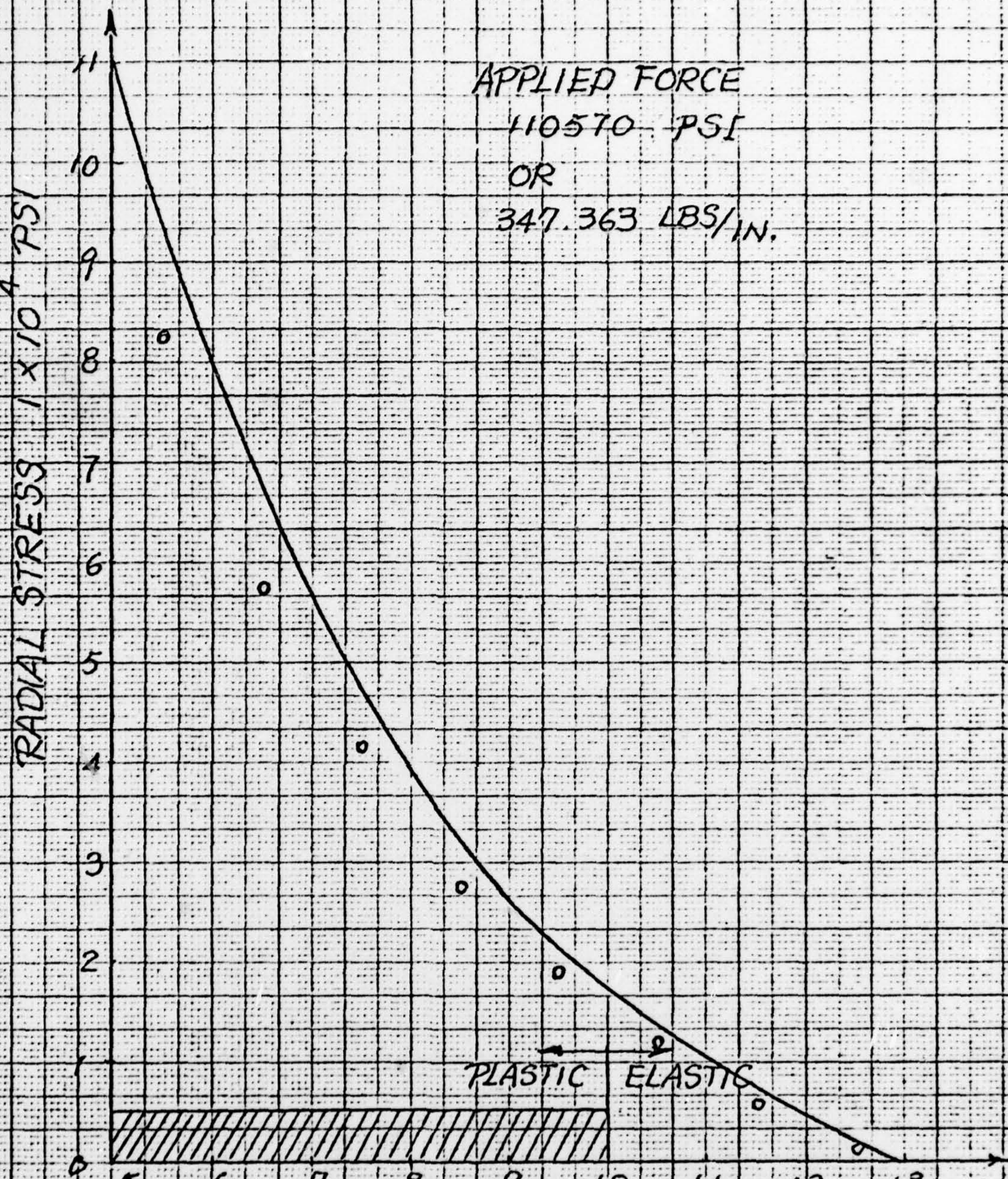


Figure 14. RADIAL STRESS VS WALL THICKNESS, THICK WALL CYLINDER, AT 110,570 PSI APPLIED FORCE.

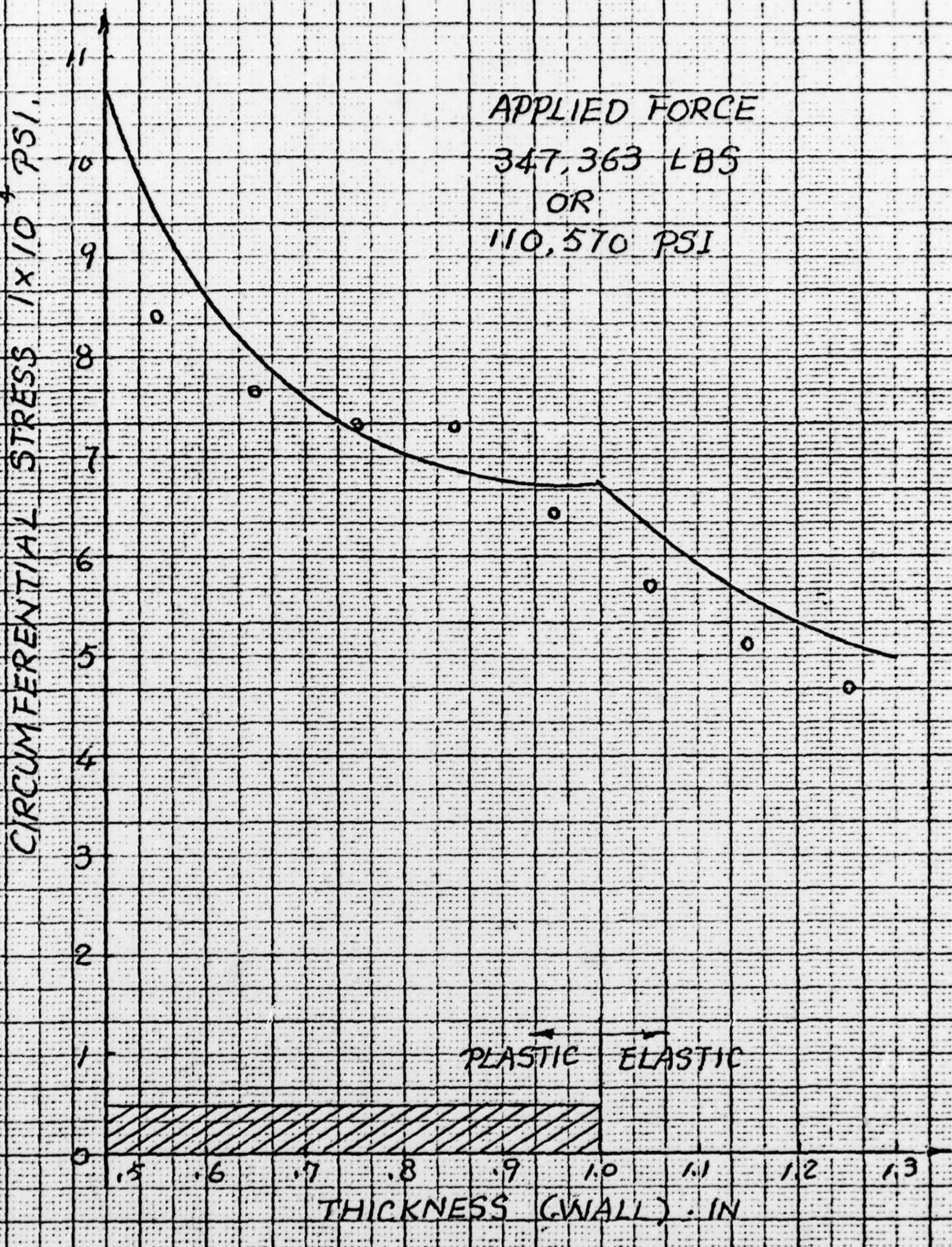


Figure 15. CIRCUMFERENTIAL STRESS VS WALL THICKNESS, THICK WALL CYLINDER, AT 110,570 PSI APPLIED FORCE.

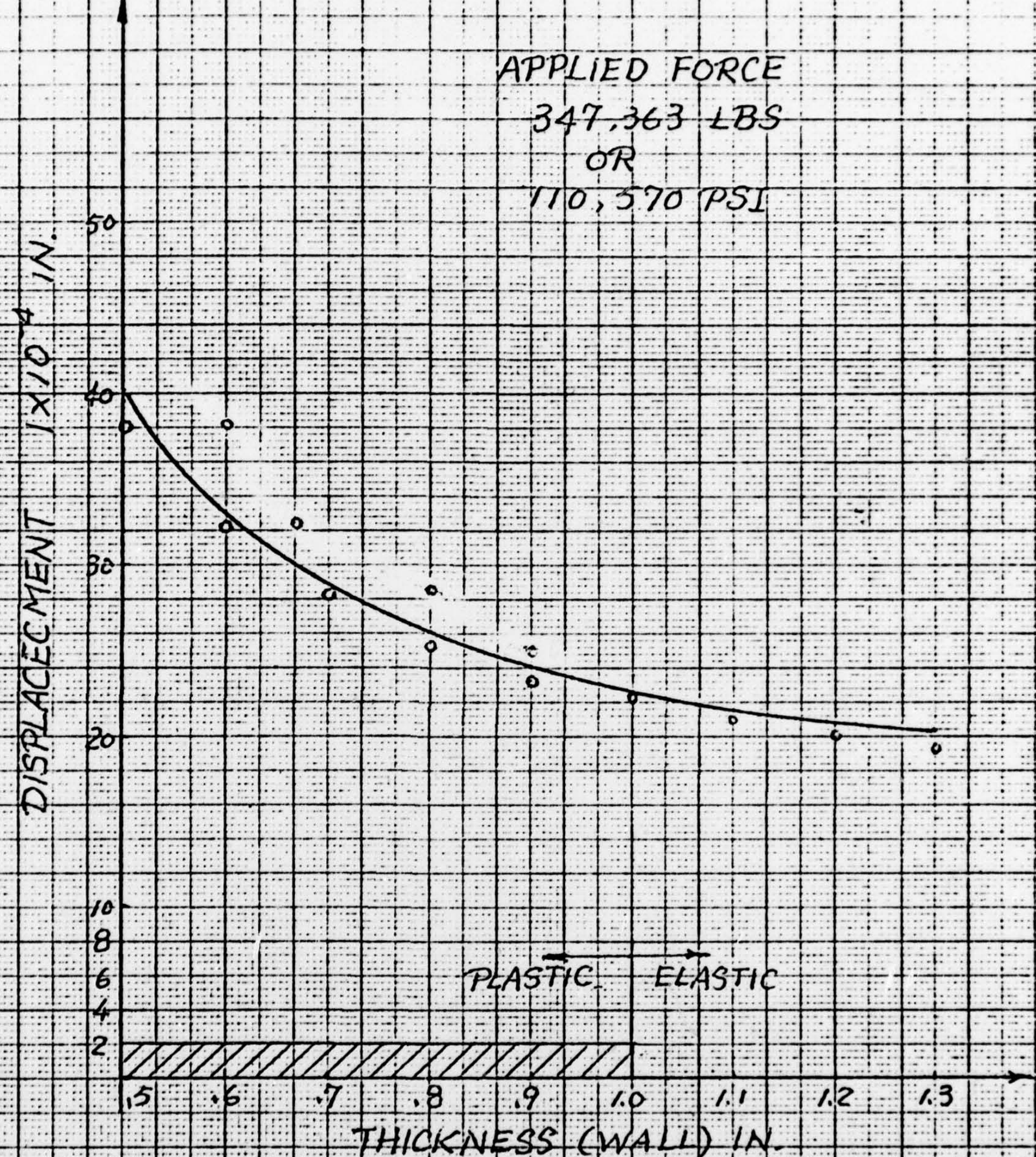
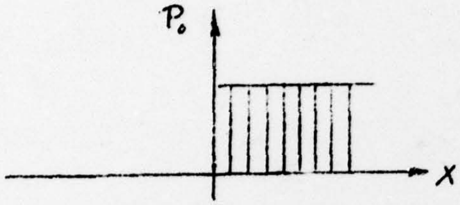


Fig 16. DISPLACEMENT, THICK WALL CYLINDER, AT 110,570 PSI APPLIED FORCE.

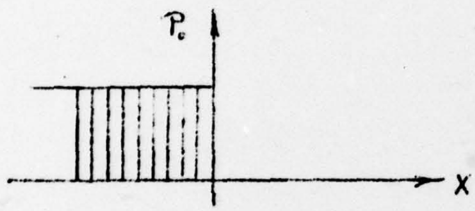
5. Pressure due to Penetration

For this presentation of the theoretical approach, a two-dimensional model is employed because of its simplicity. The basic idea is to determine the pressure distribution by matching the disturbance profile with the profile of the deformed projectile. Both these profiles are due to the interval pressure in the contact surface. To compute this pressure distribution the effects of a group of pressure element on both the projectile and the target area had to be determined.

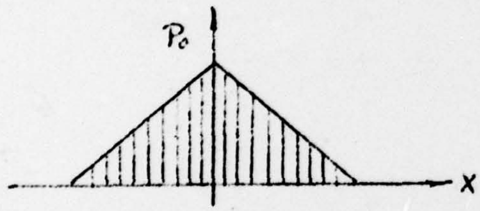
The different pressure elements are



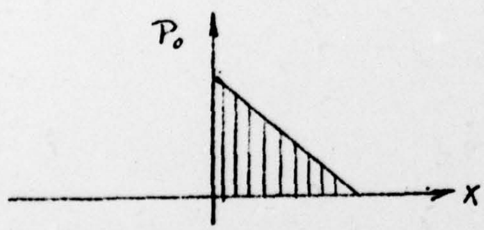
1. Aft. Semi-Infinite Pressure Band.



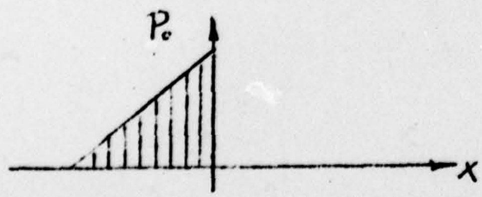
2. Forward-Infinite Pressure Band.



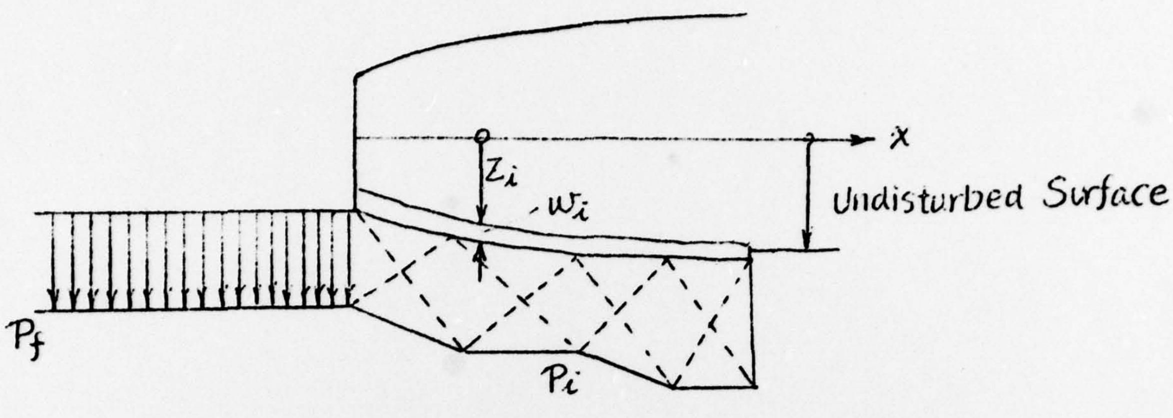
3. Complete Triangle Pressure Element.



4. Aft. Half-Triangle Pressure Element.



5. Forward Half-Triangle Pressure Element.



Pressure Distribution Diagram

From the pressure distribution diagram the solution to the problem can be obtained by summing the disturbance profiles resulting from all the pressure elements, and equating this to the profile of the deformed projectile:

$$P_f \hat{\omega}_i^{(1)} + \sum_{j=2}^{n} P_j \left(\hat{S}_{ij}^{(3)} - \hat{\omega}_{ij}^{(3)} \right) + P_f \left(\hat{S}_i^{(4)} - \hat{\omega}_i^{(4)} \right) + P_f \, \hat{S}_i^{(2)} = 0$$
$$\text{for } i = 1 \text{ to } n .$$

Here the superscript $\wedge$ means that the variable has been made dimensionless using $\varsigma \cdot g$ and the pressure for that element. The double subscript ij refers to the influence at the ith field point due to a scource element at station j. A single subscript i implies that j=1, that is, the source element is located at the leading edge. The superscripts (1) through (5) refer to the five pressure forms considered. There are a set of n equations and n unknown pressure points which can be solved by a standard matrix inversion technique.

The deformations w_i due to the five pressure forms have to be computed. This is done by standard finite element analysis.

The target deformations $_i$ due to the pressure forms are given by

Lamb 1 . Results due to the five pressure forms are listed below:

Aft Semi-Infinite Band:

$$\hat{\zeta} = \rho g \zeta / p_o = \frac{1}{\pi} \operatorname{sgn}(\lambda_o) f(\lambda_o) + H(-\lambda_o)\{2\cos(\lambda_o) - 1\}$$

where $\lambda_o = xk_o$ and $k_o = g/c^2$

in which ρ is the density, g is the acceleration of gravity, ζ is the target disturbance, c is the velocity of the travellling pressure, and x is measured forward form the point of discontinuity in the pressure. Also H is the Heaviside step function and f ix one of the two auxiliary functions for the sine-and cosine-integrals, which are defined by

$$\begin{matrix} f \\ g \end{matrix}(\lambda) = \int_0^\infty \frac{\begin{matrix} \sin \\ \cos \end{matrix}(t)}{\lambda + t} dt$$

Forward Semi-Infinite Band:

$$\hat{\zeta} = \rho g \zeta / p_o = -\frac{1}{\pi} \operatorname{sgn}(\lambda_o) f(\lambda_o) - H(-\lambda_o)\{2\cos(\lambda_o) - 1\}$$

Complete Triangular Element:

$$\begin{aligned} \hat{\zeta} = \rho g \zeta / p_o = \frac{1}{2ak_o} \Big\{ & -\left[\tfrac{1}{\pi} g\ell(\lambda_1) + H(-\lambda_1)\{2\sin(\lambda_1) - \lambda_1\}\right] \\ & + 2\left[\tfrac{1}{\pi} g\ell(\lambda_o) + H(-\lambda_o)\{2\sin(\lambda_o) - \lambda_o\}\right] \\ & - \left[\tfrac{1}{\pi} g\ell(\lambda_2) + H(-\lambda_2)\{2\sin(\lambda_2) - \lambda_2\}\right] \Big\} \end{aligned}$$

with $$gl(\lambda) = \int sgn(\lambda) f(\lambda)\, d\lambda$$
$$\lambda_1 = (x - 2a)k_o$$
$$\lambda_2 = (x + 2a)k_o$$

Aft Half-Triangular Element:

$$\hat{\varsigma} = \rho g \varsigma / p_o = \frac{1}{2ak_o} \Big\{ \Big[\frac{1}{\pi} gl(\lambda_o) + H(-\lambda_o)\{2\sin(\lambda_o) - \lambda_o\} \Big] - \Big[\frac{1}{\pi} gl(\lambda_2) + H(-\lambda_2)\{2\sin(\lambda_2) - \lambda_2\} \Big] + \Big[\frac{1}{\pi} sgn(\lambda_o) f(\lambda_o) + H(-\lambda_o)\{2\cos(\lambda_o) - 1\} \Big] \Big\}$$

Forward Half-Triangular Element:

$$\hat{\varsigma} = \rho g \varsigma / p_o = \frac{1}{2ak_o} \Big\{ \Big[\frac{1}{\pi} gl(\lambda_o) + H(-\lambda_o)\{2\sin(\lambda_o) - \lambda_o\} \Big] - \Big[\frac{1}{\pi} gl(\lambda_1) + H(-\lambda_1)\{2\sin(\lambda_1) - \lambda_1\} \Big] - \Big[\frac{1}{\pi} sgn(\lambda_o) f(\lambda_o) + H(-\lambda_o)\{2\cos(\lambda_o) - 1\} \Big] \Big\}$$

Besides the above listed deformation profiles of the target area due to the five pressure elements the deformation w_i of the projectiles due to the same pressure elements have to be computed too. For this purpose, standard finite element analysis is used with the pressure elements as load to calculate the deformation profile.

6. References

[1] Lamb: Hydrodynamics, Dover Publications, 1945.

[2] Yang and Frederick: Application of Nonlinear Analysis (PLastic) to NASTRAN using Ring Elements including Aspect Ratio Effect, Technical Note TN-1178, U.S. Army Armament Command Frankford Arsenal, Philadelphia, Penn. 19137

[3] Martin and Carey: Introduction to Finite Element Analysis, McGraw-Hill Book Company 1973.